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Global behavior of a three-dimensional linear fractional system of difference equations

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Abstract

We investigate the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \qquad y_{n+1} = \frac{c + y_n}{d + z_n}, \qquad z_{n+1} = \frac{e + z_n}{f + x_n}, \qquad n = 0, 1, \ldots,$$

where the parameters a, b, c, d, e, and f are in $(0, \infty)$ and the initial conditions x_0, y_0, and z_0 are arbitrary non-negative numbers. We obtain some global attractivity results for the positive equilibrium of this system for different values of the parameters.

Keywords: Asymptotic behavior; Difference equations; Global attractivity; Monotone

1. Introduction

Consider the following system of difference equations:

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \qquad y_{n+1} = \frac{c + y_n}{d + z_n}, \qquad z_{n+1} = \frac{e + z_n}{f + x_n}, \qquad n = 0, 1, \ldots, \tag{1}$$

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where the parameters a, b, c, d, e, and f are in $(0, \infty)$ and the initial conditions x_0, y_0, and z_0 are arbitrary non-negative numbers.

In a modelling setting, system (1) of non-linear difference equations may represent the rule by which three discrete, competitive populations reproduce from one generation to the next. The phase variables x_n, y_n, and z_n denote population sizes during the nth generation of three species A, B, and C, respectively. The sequence or orbit $\{(x_n, y_n, z_n)\colon n = 0, 1, 2, \ldots\}$ depicts how the populations evolve over time. Competition between the three populations is reflected by the fact that the transition function for each population is a decreasing function of one of the other population sizes. Competition in this model is specific in the sense that an increase in the size of species B acts to decrease the size of species A, an increase in the size of species A acts to decrease the size of species C, and an increase in the size of species C acts to decrease the size of species B. Thus, system (1) models some kind of cyclic competition. Similar systems of differential equations have been considered in [14] and [15].

Several authors have studied competitive systems such as Hassell and Comins [7], Hess [8], Franke and Yakubu [5,6], Selgrade and Ziehe [18] and Smith [19]. A simple competition model of two species that allows unbounded growth of a population size has been discussed in [1] and [2],

$$x_{n+1} = \frac{x_n}{b + y_n}, \qquad y_{n+1} = \frac{y_n}{e + x_n}, \quad n = 0, 1, \ldots. \tag{2}$$

More general system of the forms

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \qquad y_{n+1} = \frac{d + y_n}{e + x_n}, \quad n = 0, 1, \ldots, \tag{3}$$

has been investigated in [3] and [13] and

$$x_{n+1} = \frac{a + x_n}{b + cx_n + y_n}, \qquad y_{n+1} = \frac{d + y_n}{e + x_n + fy_n}, \quad n = 0, 1, \ldots, \tag{4}$$

has been considered in [12]. Related three-dimensional systems has been considered in [16] and [17] but no global attractivity result has been proved in those papers.

Here we will extend some of the two-dimensional results for (3) from [13] to the three-dimensional case of system (1) by using the monotonicity properties of the map in (1). We will also indicate how the global attractivity results for (1) can be extended to the general monotone cyclic system.

A detailed analysis of asymptotic behavior of the related simpler rational difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}},$$

has been performed in [9]. See also [11].

2. Equilibrium points

Equilibrium points $(\bar{x}, \bar{y}, \bar{z})$ of system (1) satisfy the system of equations

$$\bar{x} = \frac{a+\bar{x}}{b+\bar{y}}, \qquad \bar{y} = \frac{c+\bar{y}}{d+\bar{z}}, \qquad \bar{z} = \frac{e+\bar{z}}{f+\bar{x}}.$$

Clearly $\bar{x} \neq 0$, $\bar{y} \neq 0$ and $\bar{z} \neq 0$. Now we obtain

$$\bar{y} = \frac{a}{\bar{x}} + (1-b), \qquad \bar{z} = \frac{c}{\bar{y}} + (1-d), \qquad \bar{x} = \frac{e}{\bar{z}} + (1-f). \tag{5}$$

This implies

$$\begin{aligned}
&\bar{x} = \frac{e}{\frac{c}{\bar{y}} + (1-d)} + (1-f) \\
&\quad \Leftrightarrow \quad \bar{x} - (1-f) = \frac{e\bar{y}}{c + (1-d)\bar{y}} \\
&\quad \Leftrightarrow \quad \big[\bar{x} - (1-f)\big]\left\{c + (1-d)\left[\frac{a}{\bar{x}} + (1-b)\right]\right\} = e\left[\frac{a}{\bar{x}} + (1-b)\right] \\
&\quad \Leftrightarrow \quad \big[\bar{x} - (1-f)\big]\big\{c\bar{x} + (1-d)\big[a + (1-b)\bar{x}\big]\big\} = e\big[a + (1-b)\bar{x}\big].
\end{aligned}$$

Thus, $\bar{x}$ satisfies the quadratic equation

$$Ax^2 + Bx + C = 0, \tag{6}$$

where

$$\begin{aligned}
A &= c + (1-d)(1-b), \\
B &= (1-d)a - (1-f)c - (1-d)(1-b)(1-f) - (1-b)e, \\
C &= -a\big[e + (1-d)(1-f)\big].
\end{aligned}$$

The discriminant of Eq. (6) has the form

$$\begin{aligned}
D &= a^2(1-d)^2 + c^2(1-f)^2 + (1-b)^2\big[(1-d)(1-f) + e\big]^2 \\
&\quad + 2ac(1-d)(1-f) + 2a(1-b)(1-d)^2(1-f) \\
&\quad + 2ae(1-b)(1-d) + 2c(1-b)(1-f)\big[(1-d)(1-f) + e\big] + 4ace \\
&= \big\{a(1-d) + c(1-f) + (1-b)\big[e + (1-d)(1-f)\big]\big\}^2 + 4ace > 0.
\end{aligned}$$

The roots of Eq. (6) are given by

$$x_{\pm} = \frac{-B \pm \sqrt{D}}{2A}.$$

We are interested in a unique positive equilibrium, which is the case when (6) has exactly one positive root. Since (6) is a quadratic equation with positive discriminant, it is necessary and sufficient to have

$$\frac{C}{A} < 0$$

that is,

$$\frac{e+(1-d)(1-f)}{c+(1-d)(1-b)} > 0. \tag{7}$$

Condition (7) is satisfied if

$$b, d, f \in [1, \infty) \quad \text{or} \quad b, d, f \in (0, 1). \tag{8}$$

Similarly, condition (8) implies that system (1) has exactly one positive solution for $\bar{y}$ and $\bar{z}$, that is system (1) has a unique positive equilibrium.

Thus system (1) has a unique positive equilibrium in the following cases:

Case 2.1. $b > 1$, $d > 1$, $f > 1$;
Case 2.2. $b = 1$, $d > 1$, $f > 1$ or $d = 1$, $b > 1$, $f > 1$ or $f = 1$, $b > 1$, $d > 1$;
Case 2.3. $b = d = 1$, $f > 1$ or $d = f = 1$, $b > 1$ or $b = f = 1$, $d > 1$;
Case 2.4. $b = d = f = 1$; and
Case 2.5. $b < 1$, $d < 1$, $f < 1$.

3. Linearized stability analysis

System (1) is a special case of a general system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n, z_n), \\ y_{n+1} &= g(x_n, y_n, z_n), \\ z_{n+1} &= h(x_n, y_n, z_n), \end{aligned} \tag{9}$$

where

$$f(x, y, z) = \frac{a+x}{b+y}, \qquad g(x, y, z) = \frac{c+y}{d+z}, \qquad h(x, y, z) = \frac{e+z}{f+x}.$$

To determine the local stability of the equilibrium point $E = (\bar{x}, \bar{y}, \bar{z})$ of (1), we calculate the Jacobian of the corresponding map

$$T(x, y, z) = \left(\frac{a+x}{b+y}, \frac{c+y}{d+z}, \frac{e+z}{f+x}\right)$$

at E.

We obtain

$$J_T(E) = \begin{bmatrix} \frac{1}{b+\bar{y}} & -\frac{\bar{x}}{b+\bar{y}} & 0 \\ 0 & \frac{1}{d+\bar{z}} & -\frac{\bar{y}}{d+\bar{z}} \\ -\frac{\bar{z}}{f+\bar{x}} & 0 & \frac{1}{f+\bar{x}} \end{bmatrix}.$$

The characteristic equation of the Jacobian is

$$\lambda^3 - r\lambda^2 - s\lambda - t = 0,$$

where

$$
\begin{aligned}
r &= f_x(\bar{x},\bar{y},\bar{z}) + g_y(\bar{x},\bar{y},\bar{z}) + h_z(\bar{x},\bar{y},\bar{z}) = \frac{1}{b+\bar{y}} + \frac{1}{d+\bar{z}} + \frac{1}{f+\bar{x}},\\
s &= -f_x g_y - f_x h_z - g_y h_z + g_z h_y + g_x f_y + f_z h_x\\
&= -\frac{1}{(b+\bar{y})(d+\bar{z})} - \frac{1}{(b+\bar{y})(f+\bar{x})} - \frac{1}{(d+\bar{z})(f+\bar{x})}, \quad \text{and}\\
t &= f_x g_y h_z - f_x g_z h_y - f_y g_x h_z + f_y g_z h_x + f_z g_x h_y - f_z g_y h_x\\
&= \frac{1}{(b+\bar{y})(d+\bar{z})(f+\bar{x})}(1 - \bar{x}\bar{y}\bar{z}).
\end{aligned}
$$

The conditions for local asymptotic stability at $E = (\bar{x},\bar{y},\bar{z})$ are given by the following theorem, see [4,10]:

Theorem 3.1.

(a) *If all solutions of equation*

$$\lambda^3 - r\lambda^2 - s\lambda - t = 0 \tag{10}$$

lie inside the open disk $|\lambda| < 1$*, then the equilibrium* $(\bar{x},\bar{y},\bar{z})$ *is locally asymptotically stable.*

(b) *If at least one solution of* (10) *lies outside the closed unit disk, then the equilibrium* $(\bar{x},\bar{y},\bar{z})$ *is unstable.*

(c) *A necessary and sufficient condition for all solutions of Eq.* (10) *to lie inside the open unit disk is*

$$|r+t| < 1-s, \qquad |r-3t| < 3+s, \qquad t^2 - s - rt < 1$$

that is,

$$|\lambda_{1,2,3}| < 1 \quad \Longleftrightarrow \quad \begin{cases} |r+t| < 1-s,\\ |r-3t| < 3+s,\\ t^2 - s - rt < 1. \end{cases} \tag{11}$$

Let us check the conditions (11):

(**1**) The condition $|r+t| < 1-s$ is equivalent to

$$
\begin{aligned}
&\left|\frac{1}{b+\bar{y}} + \frac{1}{d+\bar{z}} + \frac{1}{f+\bar{x}} + \frac{(1-\bar{x}\bar{y}\bar{z})}{(b+\bar{y})(d+\bar{z})(f+\bar{x})}\right|\\
&\quad < 1 + \frac{1}{(b+\bar{y})(d+\bar{z})} + \frac{1}{(b+\bar{y})(f+\bar{x})} + \frac{1}{(d+\bar{z})(f+\bar{x})}.
\end{aligned}
$$

First, we have

$$
\begin{aligned}
&\frac{1}{b+\bar{y}} + \frac{1}{d+\bar{z}} + \frac{1}{f+\bar{x}} + \frac{(1-\bar{x}\bar{y}\bar{z})}{(b+\bar{y})(d+\bar{z})(f+\bar{x})}\\
&\quad < 1 + \frac{1}{(b+\bar{y})(d+\bar{z})} + \frac{1}{(b+\bar{y})(f+\bar{x})} + \frac{1}{(d+\bar{z})(f+\bar{x})}
\end{aligned}
$$

$$\begin{aligned}
&\Leftrightarrow \quad \begin{cases} (d+\bar z)(f+\bar x)+(b+\bar y)(f+\bar x)+(b+\bar y)(d+\bar z)+1-\bar x\bar y\bar z \\ \quad <(b+\bar y)(d+\bar z)(f+\bar x)+(f+\bar x)+(d+\bar z)+(b+\bar y) \end{cases}\\
&\Leftrightarrow \quad \begin{cases} (1-b-\bar y)(d+\bar z)(f+\bar x)+(1-b-\bar y)-(f+\bar x)(1-b-\bar y) \\ \quad -(d+\bar z)(1-b-\bar y)<\bar x\bar y\bar z \end{cases}\\
&\Leftrightarrow \quad (1-b-\bar y)(1-d-\bar z)-(1-b-\bar y)(1-d-\bar z)(f+\bar x)<\bar x\bar y\bar z\\
&\Leftrightarrow \quad (1-b-\bar y)(1-d-\bar z)(1-f-\bar x)<\bar x\bar y\bar z\\
&\Leftrightarrow \quad \frac{-a}{\bar x}\cdot\frac{-c}{\bar y}\cdot\frac{-e}{\bar z}<\bar x\bar y\bar z\\
&\Leftrightarrow \quad -ace<(\bar x\bar y\bar z)^2,
\end{aligned}$$

which is always satisfied.

Next, the condition

$$\frac{1}{b+\bar y}+\frac{1}{d+\bar z}+\frac{1}{f+\bar x}+\frac{(1-\bar x\bar y\bar z)}{(b+\bar y)(d+\bar z)(f+\bar x)} > -1-\frac{1}{(b+\bar y)(d+\bar z)}-\frac{1}{(b+\bar y)(f+\bar x)}-\frac{1}{(d+\bar z)(f+\bar x)}$$

is equivalent to

$$\begin{aligned}
&\begin{cases} (d+\bar z)(f+\bar x)+(b+\bar y)(f+\bar x)+(b+\bar y)(d+\bar z)+1-\bar x\bar y\bar z \\ \quad >-(b+\bar y)(d+\bar z)(f+\bar x)-(f+\bar x)-(d+\bar z)-(b+\bar y) \end{cases}\\
&\Leftrightarrow \quad \begin{cases} \bar x\bar y\bar z<(1+b+\bar y)(d+\bar z)(f+\bar x)+(1+b+\bar y) \\ \qquad +(f+\bar x)(1+b+\bar y)+(d+\bar z)(1+b+\bar y) \end{cases}\\
&\Leftrightarrow \quad \bar x\bar y\bar z<(1+b+\bar y)(1+d+\bar z)(f+\bar x)+(1+b+\bar y)(1+d+\bar z)\\
&\Leftrightarrow \quad \bar x\bar y\bar z<(1+f+\bar x)(1+b+\bar y)(1+d+\bar z),
\end{aligned}$$

which is always satisfied.

(2) The condition $|r-3t|<3+s \Leftrightarrow$ is equivalent to

$$\left|\frac{1}{b+\bar y}+\frac{1}{d+\bar z}+\frac{1}{f+\bar x}+\frac{3\bar x\bar y\bar z-3}{(b+\bar y)(d+\bar z)(f+\bar x)}\right| < 3-\frac{1}{(b+\bar y)(d+\bar z)}-\frac{1}{(b+\bar y)(f+\bar x)}-\frac{1}{(d+\bar z)(f+\bar x)}$$

(i)

$$\begin{aligned}
&\frac{1}{b+\bar y}+\frac{1}{d+\bar z}+\frac{1}{f+\bar x}+\frac{3\bar x\bar y\bar z-3}{(b+\bar y)(d+\bar z)(f+\bar x)}\\
&\quad <3-\frac{1}{(b+\bar y)(d+\bar z)}-\frac{1}{(b+\bar y)(f+\bar x)}-\frac{1}{(d+\bar z)(f+\bar x)}\\
&\Leftrightarrow \quad \begin{cases} 3\bar x\bar y\bar z-3<3(b+\bar y)(d+\bar z)(f+\bar x)-(d+\bar z)(f+\bar x) \\ \qquad -(b+\bar y)(f+\bar x)-(b+\bar y)(d+\bar z)-(f+\bar x) \\ \qquad -(b+\bar y)-(d+\bar z) \end{cases}
\end{aligned}$$

$$\Leftrightarrow \quad \begin{cases} 3\bar{x}\bar{y}\bar{z} < (\bar{y}+b-1)(d+\bar{z})(f+\bar{x}) + (\bar{z}+d-1)(f+\bar{x})(b+\bar{y}) \\ \qquad + (\bar{x}+f-1)(b+\bar{y})(d+\bar{z}) - (\bar{x}+f-1) - (\bar{y}+b-1) \\ \qquad - (\bar{z}+d-1). \end{cases}$$

The last condition is equivalent to

$$\left.\begin{aligned} 3\bar{x}\bar{y}\bar{z} < (\bar{x}+f-1)&[(b+\bar{y})(d+\bar{z})-1] \\ +(\bar{y}+b-1)&[(d+\bar{z})(f+\bar{x})-1] \\ +(\bar{z}+d-1)&[(f+\bar{x})(b+\bar{y})-1] \end{aligned}\right\}. \tag{12}$$

The last inequality is satisfied if $b \geqslant 1$, $d \geqslant 1$, and $f \geqslant 1$. Indeed,

$$\begin{aligned} &0 < bd-1+b\bar{z}+d\bar{y} \quad \Leftrightarrow \quad \bar{y}\bar{z} < bd-1+b\bar{z}+d\bar{y}+\bar{y}\bar{z} \\ &\quad \Leftrightarrow \quad \bar{y}\bar{z} < (b+\bar{y})(d+\bar{z})-1 \\ &\quad \Leftrightarrow \quad \bar{x}\bar{y}\bar{z} < \bar{x}\big[(b+\bar{y})(d+\bar{z})-1\big] \leqslant (\bar{x}+f-1)\big[(b+\bar{y})(d+\bar{z})-1\big]. \end{aligned}$$

We obtain the following inequalities:

$$\begin{aligned} &\bar{x}\bar{y}\bar{z} < (\bar{x}+f-1)\big[(b+\bar{y})(d+\bar{z})-1\big], \\ &\bar{x}\bar{y}\bar{z} < (\bar{y}+b-1)\big[(d+\bar{z})(f+\bar{x})-1\big], \\ &\bar{x}\bar{y}\bar{z} < (\bar{z}+d-1)\big[(f+\bar{x})(b+\bar{y})-1\big]. \end{aligned}$$

Adding these inequalities, we obtain condition (12).

Condition (12) can be simplified:

$$\begin{aligned} &3\bar{x}\bar{y}\bar{z} < \frac{a}{\bar{x}}(d+\bar{z})(f+\bar{x}) + \frac{c}{\bar{y}}(f+\bar{x})(b+\bar{y}) + \frac{e}{\bar{z}}(b+\bar{y})(d+\bar{z}) - \frac{a}{\bar{x}} - \frac{e}{\bar{z}} - \frac{c}{\bar{y}} \\ &\quad \Leftrightarrow \quad 3\bar{x}\bar{y}\bar{z} < \frac{c+\bar{y}}{\bar{y}}\frac{e+\bar{z}}{\bar{z}}\frac{a}{\bar{x}} + \frac{a+\bar{x}}{\bar{x}}\frac{e+\bar{z}}{\bar{z}}\frac{c}{\bar{y}} + \frac{a+\bar{x}}{\bar{x}}\frac{c+\bar{y}}{\bar{y}}\frac{e}{\bar{z}} - \frac{a}{\bar{x}} - \frac{e}{\bar{z}} - \frac{c}{\bar{y}} \\ &\quad \Leftrightarrow \quad 3(\bar{x}\bar{y}\bar{z})^2 < (c+\bar{y})(e+\bar{z})a + (a+\bar{x})(e+\bar{z})c + (a+\bar{x})(c+\bar{y})e \\ &\qquad\qquad - a\bar{y}\bar{z} - e\bar{x}\bar{y} - c\bar{x}\bar{z}, \end{aligned}$$

to give

$$3(\bar{x}\bar{y}\bar{z})^2 < 3ace + 2ac\bar{z} + 2ae\bar{y} + 2ce\bar{x}. \tag{13}$$

(ii) The condition

$$\begin{aligned} &\frac{1}{b+\bar{y}} + \frac{1}{d+\bar{z}} + \frac{1}{f+\bar{x}} + \frac{3\bar{x}\bar{y}\bar{z}-3}{(b+\bar{y})(d+\bar{z})(f+\bar{x})} \\ &\quad > -3 + \frac{1}{(b+\bar{y})(d+\bar{z})} + \frac{1}{(b+\bar{y})(f+\bar{x})} + \frac{1}{(d+\bar{z})(f+\bar{x})} \end{aligned}$$

is equivalent to

$$\begin{cases} (d+\bar{z})(f+\bar{x}) + (b+\bar{y})(f+\bar{x}) + (b+\bar{y})(d+\bar{z}) + 3\bar{x}\bar{y}\bar{z} - 3 \\ \quad > -3(b+\bar{y})(d+\bar{z})(f+\bar{x}) + (f+\bar{x}) + (d+\bar{z}) + (b+\bar{y}) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 3\bar{x}\bar{y}\bar{z} > & -(d+\bar{z})(f+\bar{x})(b+\bar{y}+1)-(d+\bar{z}+1)(f+\bar{x})(b+\bar{y}) \\ & -(b+\bar{y})(d+\bar{z})(f+\bar{x}+1)+(f+\bar{x}+1)+(d+\bar{z}+1) \\ & +(b+\bar{y}+1) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 3\bar{x}\bar{y}\bar{z} > (b+\bar{y}+1)[1-(d+\bar{z})(f+\bar{x})] \\ \qquad +(d+\bar{z}+1)[1-(f+\bar{x})(b+\bar{y})] \\ \qquad +(f+\bar{x}+1)[1-(b+\bar{y})(d+\bar{z})]. \end{cases}$$

The last inequality is always satisfied since the right-hand side is always negative, i.e.,

$$1-(f+\bar{x})(b+\bar{y})<0 \quad \Leftrightarrow \quad 1<\left(\frac{e}{\bar{z}}+1\right)\left(\frac{a}{\bar{x}}+1\right).$$

(**3**) The condition $t^2-s-rt<1$ is equivalent to

$$\frac{(1-\bar{x}\bar{y}\bar{z})^2}{(b+\bar{y})^2(d+\bar{z})^2(f+\bar{x})^2}+\frac{1}{(b+\bar{y})(d+\bar{z})}+\frac{1}{(b+\bar{y})(f+\bar{x})}+\frac{1}{(d+\bar{z})(f+\bar{x})}$$
$$-\frac{\left(\frac{1}{b+\bar{y}}+\frac{1}{d+\bar{z}}+\frac{1}{f+\bar{x}}\right)(1-\bar{x}\bar{y}\bar{z})}{(b+\bar{y})(d+\bar{z})(f+\bar{x})}<1$$

$$\Leftrightarrow \quad \begin{cases} (1-\bar{x}\bar{y}\bar{z})^2+(b+\bar{y})(d+\bar{z})(f+\bar{x})^2+(b+\bar{y})(d+\bar{z})^2(f+\bar{x}) \\ \quad +(b+\bar{y})^2(d+\bar{z})(f+\bar{x})-(d+\bar{z})(f+\bar{x})-(b+\bar{y})(f+\bar{x}) \\ \quad -(b+\bar{y})(d+\bar{z})+(d+\bar{z})(f+\bar{x})\bar{x}\bar{y}\bar{z}+(b+\bar{y})(f+\bar{x})\bar{x}\bar{y}\bar{z} \\ \quad +(b+\bar{y})(d+\bar{z})\bar{x}\bar{y}\bar{z}<(b+\bar{y})^2(d+\bar{z})^2(f+\bar{x})^2 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} (1-\bar{x}\bar{y}\bar{z})^2+(d+\bar{z})(f+\bar{x})[(b+\bar{y})(f+\bar{x})-1] \\ \quad +(b+\bar{y})(f+\bar{x})[(b+\bar{y})(d+\bar{z})-1] \\ \quad +(b+\bar{y})(d+\bar{z})[(d+\bar{z})(f+\bar{x})-1] \\ \quad +(d+\bar{z})(f+\bar{x})\bar{x}\bar{y}\bar{z}+(b+\bar{y})(f+\bar{x})\bar{x}\bar{y}\bar{z} \\ \quad +(b+\bar{y})(d+\bar{z})\bar{x}\bar{y}\bar{z}<(b+\bar{y})^2(d+\bar{z})^2(f+\bar{x})^2. \end{cases}$$

This is equivalent to

$$\left.\begin{array}{l} (1-\bar{x}\bar{y}\bar{z})^2+(d+\bar{z})(f+\bar{x})[(b+\bar{y})(f+\bar{x})-1+\bar{x}\bar{y}\bar{z}] \\ \quad +(b+\bar{y})(f+\bar{x})[(b+\bar{y})(d+\bar{z})-1+\bar{x}\bar{y}\bar{z}] \\ \quad +(b+\bar{y})(d+\bar{z})[(d+\bar{z})(f+\bar{x})-1+\bar{x}\bar{y}\bar{z}] \\ \quad <(b+\bar{y})^2(d+\bar{z})^2(f+\bar{x})^2 \end{array}\right\}. \tag{14}$$

Thus, the equilibrium $(\bar{x},\bar{y},\bar{z})$ is locally asymptotically stable if conditions (14) and either (12) or the equivalent condition (13) are satisfied. This can be formulated as

Theorem 3.2. *The equilibrium* $(\bar{x},\bar{y},\bar{z})$ *of system* (1) *is locally asymptotically stable if the condition* (14) *and either* (12) *or* (13) *are satisfied.*

In particular, the equilibrium $(\bar{x},\bar{y},\bar{z})$ *of system* (1) *is locally asymptotically stable if*

$$b\geqslant 1, \qquad d\geqslant 1, \qquad f\geqslant 1. \tag{15}$$

Proof. We will show that (15) implies (14). Dropping all bars condition (14) is equivalent to:

$$
\begin{aligned}
&(b+y)^2(d+z)^2(f+x)^2-(1-xyz)^2\\
&\quad-(d+z)(f+x)\big((b+y)(f+x)-1+xyz\big)\\
&\quad-(b+y)(f+x)\big((b+y)(d+z)-1+xyz\big)\\
&\quad-(b+y)(d+z)\big((d+z)(f+x)-1+xyz\big)>0.
\end{aligned}
$$

Collecting similar terms, we obtain

$$
\begin{aligned}
&y^2z^2f^2+b^2z^2x^2+y^2d^2x^2+xyz+byd^2x^2+byz^2x^2+bydzx^2+y^2d^2fx\\
&\quad+y^2dzf^2+b^2z^2fx+y^2z^2fx+yz^2fx+y^2dzfx+y^2dzx^2+b^2dzx^2\\
&\quad+byz^2f^2+(bd-1)(fd-1)(bf-1)+yx^2d(bd-1)+x^2yz^2(b-1)\\
&\quad+yx^2z(bd-1)+2bdzx(bf-1)+2bdyx(df-1)+y^2dx(df-1)\\
&\quad+y^2zf(df-1)+bz^2x(bf-1)+xyzbf(d-1)+xyzbd(f-1)\\
&\quad+xyzb(df-1)+2xyzd(bf-1)+2xyzf(bd-1)+bx^2d(bd-1)\\
&\quad+bz^2f(bf-1)+y^2z^2x(f-1)+xyz^2f(b-1)+x^2yzb(d-1)\\
&\quad+yz^2x(bf-1)+bz^2xy(f-1)+y^2df(df-1)+y^2zx(df-1)\\
&\quad+y^2zxf(d-1)+y^2zxd(f-1)+x^2y^2z(d-1)+bx^2z(bd-1)\\
&\quad+yz^2f(bf-1)+2ydzf(bf-1)+x^2yzd(b-1)+xyz(df-1)(b-1)\\
&\quad+yf(df-1)(bd-1)+bz(bf-1)(df-1)+zf(bf-1)(bd-1)\\
&\quad+dy(df-1)(bf-1)+bx(bd-1)(fd-1)+dx(bd-1)(bf-1)\\
&\quad+xz(1-b^2+2b^2fd-2bf)+xy(1-d^2+2bfd^2-2fd)\\
&\quad+zy(1-f^2+2bdf^2-2bf)>0.
\end{aligned}
$$

Clearly, all terms in the above sum are either positive or non-negative with the exception of the last three terms. Now, we will show that the last three terms are non-negative if condition (15) is satisfied

$$
\begin{aligned}
&xz(1-b^2+2b^2fd-2bf)\\
&\quad\geqslant xz(1-b^2+2b^2f-2bf)=xz\big[1-b^2+2bf(b-1)\big]\\
&\quad=xz(b-1)\big(2bf-(b+1)\big)\geqslant xz(b-1)(2b-b-1)=xz(b-1)^2\geqslant 0,
\end{aligned}
$$

$$
\begin{aligned}
&xy(1-d^2+2bfd^2-2fd)\\
&\quad\geqslant xy(1-d^2+2fd^2-2fd)=xy\big[1-d^2+2fd(d-1)\big]\\
&\quad=xy(d-1)(2fd-d-1)\geqslant xy(d-1)(2d-d-1)=xy(d-1)^2\geqslant 0,
\end{aligned}
$$

$$
\begin{aligned}
&zy(1-f^2+2bdf^2-2bf)\\
&\quad\geqslant zy(1-f^2+2bf^2-2bf)=zy(f-1)(2bf-f-1)\\
&\quad\geqslant zy(f-1)[2f-f-1]=zy(f-1)^2\geqslant 0. \qquad\square
\end{aligned}
$$

4. Global attractivity results

In this section we present global attractivity results for the general system (9) that can be used to prove global attractivity of the equilibrium of system (1). Similar results for two-dimensional systems have been obtained in [12] and [13] and for second-order difference equations in [9] and references mentioned therein.

Theorem 4.1. *Let $[a_1, b_1]$, $[a_2, b_2]$, and $[a_3, b_3]$ be intervals such that*

$$f : \mathcal{B} \to [a_1, b_1], \qquad g : \mathcal{B} \to [a_2, b_2], \qquad h : \mathcal{B} \to [a_3, b_3],$$

where $\mathcal{B} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, and $f, g,$ and h are continuous functions that satisfy the following conditions:

(a) $f(x, y, z)$ *is non-decreasing in x and z and non-increasing in y for every $(x, y, z) \in \mathcal{B}$;*
$g(x, y, z)$ *is non-decreasing in x and y and non-increasing in z for every $(x, y, z) \in \mathcal{B}$;*
$h(x, y, z)$ *is non-decreasing in y and z and non-increasing in x for every $(x, y, z) \in \mathcal{B}$.*
(b) *If the system*

$$\begin{cases} m_1 = f(m_1, M_2, m_3), & M_1 = f(M_1, m_2, M_3), \\ m_2 = g(m_1, m_2, M_3), & M_2 = g(M_1, M_2, m_3), \\ m_3 = h(M_1, m_2, m_3), & M_3 = h(m_1, M_2, M_3), \end{cases} \tag{16}$$

has a solution, then $m_1 = M_1$, $m_2 = M_2$ and $m_3 = M_3$.

Then every solution of Eq. (9) *with $(x_0, y_0, z_0) \in \mathcal{B}$ converges to the equilibrium $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}$.*

Proof. Set

$$m_1^{(0)} = a_1, \qquad m_2^{(0)} = a_2, \qquad m_3^{(0)} = a_3,$$
$$M_1^{(0)} = b_1, \qquad M_2^{(0)} = b_2, \qquad M_3^{(0)} = b_3$$

and for $i = 1, 2, 3, \ldots$ define

$$\begin{cases} m_1^{(i)} = f(m_1^{(i-1)}, M_2^{(i-1)}, m_3^{(i-1)}), & M_1^{(i)} = f(M_1^{(i-1)}, m_2^{(i-1)}, M_3^{(i-1)}), \\ m_2^{(i)} = g(m_1^{(i-1)}, m_2^{(i-1)}, M_3^{(i-1)}), & M_2^{(i)} = g(M_1^{(i-1)}, M_2^{(i-1)}, m_3^{(i-1)}), \\ m_3^{(i)} = h(M_1^{(i-1)}, m_2^{(i-1)}, m_3^{(i-1)}), & M_3^{(i)} = h(m_1^{(i-1)}, M_2^{(i-1)}, M_3^{(i-1)}). \end{cases} \tag{17}$$

Using the invariance of the box $\mathcal{B}$, we obtain

$$m_1^{(0)} \leqslant m_1^{(1)}, \qquad m_2^{(0)} \leqslant m_2^{(1)}, \qquad m_3^{(0)} \leqslant m_3^{(1)},$$
$$M_1^{(0)} \geqslant M_1^{(1)}, \qquad M_2^{(0)} \geqslant M_2^{(1)}, \qquad M_3^{(0)} \geqslant M_3^{(1)}.$$

Using the monotonicity of f, g, and h, we have

$$m_1^{(1)} = f\big(m_1^{(0)}, M_2^{(0)}, m_3^{(0)}\big) \leqslant f\big(m_1^{(1)}, M_2^{(1)}, m_3^{(1)}\big) = m_1^{(2)},$$
$$m_2^{(1)} = g\big(m_1^{(0)}, m_2^{(0)}, M_3^{(0)}\big) \leqslant g\big(m_1^{(1)}, m_2^{(1)}, M_3^{(1)}\big) = m_2^{(2)},$$

$$m_3^{(1)} = h\big(M_1^{(0)}, m_2^{(0)}, m_3^{(0)}\big) \leqslant h\big(M_1^{(1)}, m_2^{(1)}, m_3^{(1)}\big) = m_3^{(2)},$$
$$M_1^{(1)} = f\big(M_1^{(0)}, m_2^{(0)}, M_3^{(0)}\big) \geqslant f\big(M_1^{(1)}, m_2^{(1)}, M_3^{(1)}\big) = M_1^{(2)},$$
$$M_2^{(1)} = g\big(M_1^{(0)}, M_2^{(0)}, m_3^{(0)}\big) \geqslant g\big(M_1^{(1)}, M_2^{(1)}, m_3^{(1)}\big) = M_2^{(2)},$$
$$M_3^{(1)} = h\big(m_1^{(0)}, M_2^{(0)}, M_3^{(0)}\big) \geqslant h\big(m_1^{(1)}, M_2^{(1)}, M_3^{(1)}\big) = M_3^{(2)}.$$

By induction, we obtain for $i = 0, 1, 2, \ldots,$

$$a_1 = m_1^{(0)} \leqslant m_1^{(1)} \leqslant \cdots \leqslant m_1^{(i)} \leqslant \cdots \leqslant M_1^{(i)} \leqslant \cdots \leqslant M_1^{(1)} \leqslant M_1^{(0)} = b_1,$$
$$a_2 = m_2^{(0)} \leqslant m_2^{(1)} \leqslant \cdots \leqslant m_2^{(i)} \leqslant \cdots \leqslant M_2^{(i)} \leqslant \cdots \leqslant M_2^{(1)} \leqslant M_2^{(0)} = b_2,$$
$$a_3 = m_3^{(0)} \leqslant m_3^{(1)} \leqslant \cdots \leqslant m_3^{(i)} \leqslant \cdots \leqslant M_3^{(i)} \leqslant \cdots \leqslant M_3^{(1)} \leqslant M_3^{(0)} = b_3.$$

Clearly

$$m_1^{(0)} \leqslant x_0 \leqslant M_1^{(0)},$$
$$m_2^{(0)} \leqslant y_0 \leqslant M_2^{(0)},$$
$$m_3^{(0)} \leqslant z_0 \leqslant M_3^{(0)},$$

which by the monotonicity of f, g, and h implies

$$m_1^{(1)} = f\big(m_1^{(0)}, M_2^{(0)}, m_3^{(0)}\big) \leqslant x_1 = f(x_0, y_0, z_0) \leqslant f\big(M_1^{(0)}, m_2^{(0)}, M_3^{(0)}\big) = M_1^{(1)},$$
$$m_2^{(1)} = g\big(m_1^{(0)}, m_2^{(0)}, M_3^{(0)}\big) \leqslant y_1 = g(x_0, y_0, z_0) \leqslant g\big(M_1^{(0)}, M_2^{(0)}, m_3^{(0)}\big) = M_2^{(1)},$$
$$m_3^{(1)} = h\big(M_1^{(0)}, m_2^{(0)}, m_3^{(0)}\big) \leqslant z_1 = h(x_0, y_0, z_0) \leqslant h\big(m_1^{(0)}, M_2^{(0)}, M_3^{(0)}\big) = M_3^{(1)},$$
$$m_1^{(2)} = f\big(m_1^{(1)}, M_2^{(1)}, m_3^{(1)}\big) \leqslant x_2 = f(x_1, y_1, z_1) \leqslant f\big(M_1^{(1)}, m_2^{(1)}, M_3^{(1)}\big) = M_1^{(2)},$$
$$m_2^{(2)} = g\big(m_1^{(1)}, m_2^{(1)}, M_3^{(1)}\big) \leqslant y_2 = g(x_1, y_1, z_1) \leqslant g\big(M_1^{(1)}, M_2^{(1)}, m_3^{(1)}\big) = M_2^{(2)},$$
$$m_3^{(2)} = h\big(M_1^{(1)}, m_2^{(1)}, m_3^{(1)}\big) \leqslant z_2 = h(x_1, y_1, z_1) \leqslant h\big(m_1^{(1)}, M_2^{(1)}, M_3^{(1)}\big) = M_3^{(2)}.$$

By induction, it follows that

$$m_1^{(k)} \leqslant x_k \leqslant M_1^{(k)},$$
$$m_2^{(k)} \leqslant y_k \leqslant M_2^{(k)},$$
$$m_3^{(k)} \leqslant z_k \leqslant M_3^{(k)},$$

for every $k = 0, 1, 2, \ldots.$ Thus, there exist numbers m_1, m_2, m_3, M_1, M_2, and M_3 such that

$$m_1 = \lim_{k\to\infty} m_1^{(k)}, \qquad m_2 = \lim_{k\to\infty} m_2^{(k)}, \qquad m_3 = \lim_{k\to\infty} m_3^{(k)},$$
$$M_1 = \lim_{k\to\infty} M_1^{(k)}, \qquad M_2 = \lim_{k\to\infty} M_2^{(k)}, \qquad M_3 = \lim_{k\to\infty} M_3^{(k)}.$$

Clearly

$$m_1 \leqslant \varliminf_{k\to\infty} x_k \leqslant \varlimsup_{k\to\infty} x_k \leqslant M_1,$$
$$m_2 \leqslant \varliminf_{k\to\infty} y_k \leqslant \varlimsup_{k\to\infty} y_k \leqslant M_2,$$
$$m_3 \leqslant \varliminf_{k\to\infty} z_k \leqslant \varlimsup_{k\to\infty} z_k \leqslant M_3.$$

Using the continuity of f, g, and h, (17) implies

$$\begin{cases} m_1 = f(m_1, M_2, m_3), & M_1 = f(M_1, m_2, M_3), \\ m_2 = g(m_1, m_2, M_3), & M_2 = g(M_1, M_2, m_3), \\ m_3 = h(M_1, m_2, m_3), & M_3 = h(m_1, M_2, M_3). \end{cases}$$

By assumption (b), we obtain

$$m_1 = M_1 = \bar{x},$$
$$m_2 = M_2 = \bar{y},$$
$$m_3 = M_3 = \bar{z},$$

which completes the proof. □

In a similar way we can prove the following result:

Theorem 4.2. *Let* $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3]$, f, g, *and* h *be the intervals and functions as in Theorem* 4.1 *that satisfy the following conditions*:

(a) $f(x, y, z)$ *is non-decreasing in* x *and non-increasing in* y *and* z *for every* $(x, y, z) \in \mathcal{B}$;
$g(x, y, z)$ *is non-decreasing in* y *and non-increasing in* x *and* z *for every* $(x, y, z) \in \mathcal{B}$;
$h(x, y, z)$ *is non-decreasing in* z *and non-increasing in* x *and* y *for every* $(x, y, z) \in \mathcal{B}$.

(b) *If the system*

$$\begin{cases} m_1 = f(m_1, M_2, M_3), & M_1 = f(M_1, m_2, m_3), \\ m_2 = g(M_1, m_2, M_3), & M_2 = g(m_1, M_2, m_3), \\ m_3 = h(M_1, M_2, m_3), & M_3 = h(m_1, m_2, M_3), \end{cases} \tag{18}$$

has a solution, then $m_1 = M_1$, $m_2 = M_2$, *and* $m_3 = M_3$.

Then every solution of Eq. (9) *which has one point* (x_n, y_n, z_n) *in* $\mathcal{B}$ *converges to the equilibrium* $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{B}$.

The box $\mathcal{B}$ introduced in Theorems 4.1 and 4.2 has the property that $(f, g, h) : \mathcal{B} \to \mathcal{B}$, i.e., $\mathcal{B}$ is an invariant box for Eq. (9).

5. Invariant boxes

In this section we will find invariant boxes and apply Theorem 4.1 to obtain either global attractivity results or attractivity results for several cases depending of some special values of the parameters.

5.1. Case $b > 1$, $d > 1$, $f > 1$

In this case, system (1) has a unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$.

System (1) can be represented in the form (9) where

$$f(x, y, z) = \frac{a + x}{b + y}, \qquad g(x, y, z) = \frac{c + y}{d + z}, \qquad h(x, y, z) = \frac{e + z}{f + x}.$$

Using the monotonic properties of functions f, g, and h, we will determine the invariant boxes for (1) by determining invariant intervals for each component x, y, and z:

$$[L_1, U_1],\ [L_2, U_2],\ [L_3, U_3],$$

where L_i, U_i, $i = 1, 2, 3$, are the bounds for x, y, and z, respectively. For simplicity, we assume that $L_i = L$, $U_i = U$ for $i = 1, 2, 3$. Now, we have

$$\begin{aligned}
L &\leqslant \frac{a + L}{b + U} \leqslant x_{n+1} = \frac{a + x_n}{b + y_n} \leqslant \frac{a + U}{b + L} \leqslant U, \\
L &\leqslant \frac{c + L}{d + U} \leqslant y_{n+1} = \frac{c + y_n}{d + z_n} \leqslant \frac{c + U}{d + L} \leqslant U, \\
L &\leqslant \frac{e + L}{f + U} \leqslant z_{n+1} = \frac{e + z_n}{f + x_n} \leqslant \frac{e + U}{f + L} \leqslant U,
\end{aligned}$$

and

$$\begin{aligned}
&a + (1 - b)U \leqslant LU \leqslant a + (1 - b)L, \\
&c + (1 - d)U \leqslant LU \leqslant c + (1 - d)L, \\
&e + (1 - f)U \leqslant LU \leqslant e + (1 - f)L.
\end{aligned}$$

These inequalities are consistent when the following condition is satisfied:

$$b \geqslant 1, \qquad d \geqslant 1, \qquad f \geqslant 1. \tag{19}$$

Taking $L = 0$ in the above inequalities, we obtain

$$U \geqslant \max\left\{\frac{a}{b - 1}, \frac{c}{d - 1}, \frac{e}{f - 1}\right\}. \tag{20}$$

Thus, the invariant box for system (1) has the form

$$S_{\mathrm{inv}} = [0, U]^3, \tag{21}$$

where U satisfies (20).

Now, we apply Theorem 4.1 to this case. Condition (b) takes the form:

$$\begin{cases}
m_1 = \frac{a+m_1}{b+M_2}, & M_1 = \frac{a+M_1}{b+m_2}, \\
m_2 = \frac{c+m_2}{d+M_3}, & M_2 = \frac{c+M_2}{d+m_3}, \\
m_3 = \frac{e+m_3}{f+M_1}, & M_3 = \frac{e+M_3}{f+m_1},
\end{cases}$$

that is,

$$\frac{m_1}{M_1} = \frac{b-1+m_2}{b-1+M_2},$$
$$\frac{m_2}{M_2} = \frac{d-1+m_3}{d-1+M_3},$$
$$\frac{m_3}{M_3} = \frac{f-1+m_1}{f-1+M_1}.$$

Setting

$$\frac{m_1}{M_1} = \alpha_1 \leqslant 1, \qquad \frac{m_2}{M_2} = \alpha_2 \leqslant 1 \quad \text{and} \quad \frac{m_3}{M_3} = \alpha_3 \leqslant 1,$$

we have

$$\begin{aligned} (b-1)(\alpha_1 - 1) &= M_2(\alpha_2 - \alpha_1), \\ (d-1)(\alpha_2 - 1) &= M_3(\alpha_3 - \alpha_2), \\ (f-1)(\alpha_3 - 1) &= M_1(\alpha_1 - \alpha_3). \end{aligned} \tag{22}$$

When $b > 1$, $d > 1$, and $f > 1$, the left-hand sides of (22) are less than or equal to zero, and thus

$$\alpha_2 - \alpha_1 \leqslant 0, \qquad \alpha_3 - \alpha_2 \leqslant 0 \quad \text{and} \quad \alpha_1 - \alpha_3 \leqslant 0.$$

This implies

$$\alpha_1 \leqslant \alpha_3 \leqslant \alpha_2 \leqslant \alpha_1,$$

which holds if and only if $\alpha_1 = \alpha_2 = \alpha_3$. In view of (22), it follows that $\alpha_1 = \alpha_2 = \alpha_3 = 1$, that is

$$m_1 = M_1, \qquad m_2 = M_2, \qquad m_3 = M_3.$$

Consequently, condition (b) of Theorem 4.1 is satisfied. Using Theorems 4.1 and 3.2 and the fact that the right-hand limit U of invariant intervals can be chosen arbitrarily large, we obtain the following global attractivity result:

Theorem 5.1. *Assume that $b > 1$, $d > 1$, and $f > 1$. Then the set S_{inv}, defined by* (21)*, where U satisfies* (20)*, is an invariant and attracting set, and the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ is a global attractor, that is*

$$\lim_{n\to\infty} x_n = \bar{x}, \qquad \lim_{n\to\infty} y_n = \bar{y}, \qquad \lim_{n\to\infty} z_n = \bar{z},$$

for every solution $\{(x_n, y_n, z_n)\}$ of system (1)*.*

Furthermore, the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ is globally asymptotically stable.

5.2. Case $b = 1$, $d > 1$, $f > 1$ ($d = 1$, $b > 1$, $f > 1$ or $f = 1$, $b > 1$, $d > 1$)

In this case system (1) has a unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$. We will consider the case $b = 1$, $d > 1$, $f > 1$. The other two cases are similar.

As in case 5.1, we obtain the invariant box:

$$R_{\text{inv}} = \left[\frac{a}{U}, U\right] \times \left[\frac{a}{U}, U\right] \times [0, U_3], \tag{23}$$

where

$$U > \max\left\{\frac{a}{c}\left(d-1+\frac{e}{f-1+\sqrt{a}}\right), \sqrt{a}\right\}, \qquad U \geqslant \frac{c}{d-1},$$
$$U_3 > \frac{e}{f-1+\sqrt{a}}. \tag{24}$$

The bounds for an invariant box are obtained from the following conditions:

$$L \leqslant \frac{a+L}{1+U} \leqslant x_{n+1} = \frac{a+x_n}{1+y_n} \leqslant \frac{a+U}{1+L} \leqslant U,$$
$$L \leqslant \frac{c+L}{d+U_3} \leqslant y_{n+1} = \frac{c+y_n}{d+z_n} \leqslant \frac{c+U}{d+L_3} \leqslant U,$$
$$L_3 \leqslant \frac{e+L_3}{f+U} \leqslant z_{n+1} = \frac{e+z_n}{f+x_n} \leqslant \frac{e+U_3}{f+L} \leqslant U_3.$$

We want to apply Theorem 4.1. First, equality in (22) implies $\alpha_1 = \alpha_2$, and by multiplication of the last two equalities in (22), we obtain

$$(d-1)(f-1)(\alpha_1-1)(\alpha_3-1) = M_1M_3(\alpha_3-\alpha_1)(\alpha_1-\alpha_3).$$

For this equality to hold we must have $\alpha_1 = 1$ or $\alpha_3 = 1$, which by condition (22), implies

$$m_1 = M_1, \qquad m_2 = M_2, \qquad m_3 = M_3.$$

Thus, condition (b) of Theorem 4.1 is satisfied. Using Theorems 4.1 and 3.2 and the fact that the right-hand limits U and U_3 of invariant intervals can be chosen arbitrarily large, making the left-hand limit a/U arbitrarily small, we obtain the following global attractivity result:

Theorem 5.2. *Assume that $b = 1$, $d > 1$, and $f > 1$. Then the set R_{inv}, defined by* (23), *where U and U_3 are given by* (24), *is an invariant and attracting set, and the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ of system* (1) *is a global attractor, that is*

$$\lim_{n\to\infty} x_n = \bar{x}, \qquad \lim_{n\to\infty} y_n = \bar{y}, \qquad \lim_{n\to\infty} z_n = \bar{z},$$

for every solution $\{(x_n, y_n, z_n)\}$ of system (1).

Furthermore, the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ of system (1) *is globally asymptotically stable.*

Remark 5.1. Cases $d = 1$, $b > 1$, $f > 1$ and $f = 1$, $b > 1$, $d > 1$ are analogous and the same result holds with invariant sets $R_{\mathrm{inv}2}$ and $R_{\mathrm{inv}3}$ defined appropriately.

5.3. Case $b = d = 1$, $f > 1$ ($d = f = 1$, $b > 1$ or $b = f = 1$, $d > 1$)

In this case system (1) has an unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$.

As in case 5.1, we obtain the invariant box to be

$$P_{\mathrm{inv}} = [0, U_1] \times \left[\frac{c}{U_2}, U_2\right] \times \left[\frac{c}{U_2}, U_2\right], \tag{25}$$

where

$$U_1 \geqslant \frac{a}{c} U_2, \qquad e > a, \qquad U_2 \geqslant \max\left\{ \frac{c(f-1)}{e-a}, \frac{e}{f-1} \right\}, \qquad U_2 > \sqrt{c}. \tag{26}$$

The bounds for an invariant box are obtained from the following conditions:

$$\begin{aligned}
L_1 &\leqslant \frac{a+L_1}{1+U_2} \leqslant x_{n+1} = \frac{a+x_n}{1+y_n} \leqslant \frac{a+U_1}{1+L_2} \leqslant U_1, \\
L_2 &\leqslant \frac{c+L_2}{1+U_2} \leqslant y_{n+1} = \frac{c+y_n}{1+z_n} \leqslant \frac{c+U_2}{1+L_2} \leqslant U_2, \\
L_2 &\leqslant \frac{e+L_2}{f+U_1} \leqslant z_{n+1} = \frac{e+z_n}{f+x_n} \leqslant \frac{e+U_2}{f+L_1} \leqslant U_2.
\end{aligned}$$

We want to apply Theorem 4.1. The first two equalities in (22) imply $\alpha_1 = \alpha_2 = \alpha_3$ and the third equality implies $\alpha_3 = 1$ that is,

$$m_1 = M_1, \qquad m_2 = M_2, \qquad m_3 = M_3.$$

Consequently, condition (b) of Theorem 4.1 is satisfied. Using Theorems 4.1 and 3.2 and the fact that the right-hand limits U_1 and U_2 of invariant intervals can be chosen arbitrarily large, in which case $\frac{c}{U_2}$ is arbitrarily small, we obtain the following global attractivity result:

Theorem 5.3. *Assume that* $b = d = 1$ *and* $f > 1$. *Then the set* P_{inv}, *defined by* (25), *where* U_1 *and* U_2 *are given by* (26), *is an invariant and attracting set, and the unique positive equilibrium* $(\bar{x}, \bar{y}, \bar{z})$ *of system* (1) *is a global attractor, that is*

$$\lim_{n\to\infty} x_n = \bar{x}, \qquad \lim_{n\to\infty} y_n = \bar{y}, \qquad \lim_{n\to\infty} z_n = \bar{z},$$

for every solution $\{(x_n, y_n, z_n)\}$ *of system* (1).

Furthermore, the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ *of system* (1) *is globally asymptotically stable.*

Remark 5.2. Cases $d = f = 1, b > 1$ and $b = f = 1, b > 1$, are analogous and the same result holds with invariant sets $P_{\text{inv}2}$ and $P_{\text{inv}3}$ defined appropriately.

Remark 5.3. General global attractivity result such as Theorem 4.1 can be obtained for the general system of k equations and then applied to the monotone cyclic system of the form:

$$x_{n+1}^i = \frac{a_i + x_n^i}{b_i + x_n^{i+1}}, \quad i = 1, 2, \dots, k, \ n = 0, 1, \dots, \ x_n^{k+1} = x_n^1, \tag{27}$$

where the parameters a_i and b_i are in $(0, \infty)$ and the initial conditions x_0^i, $i = 1, 2, \dots, k$, are arbitrary non-negative numbers, to obtain the corresponding global attractivity results. Local asymptotic stability analysis in this case is very complicated.

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